MATH 312
Lecture 23

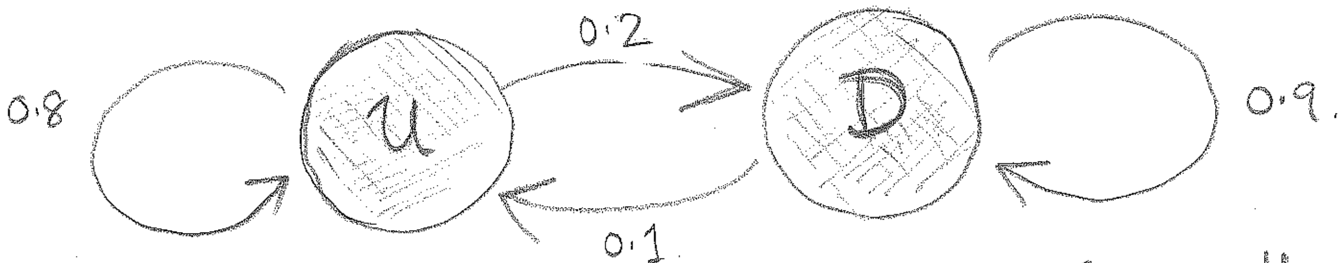
MARKOV CHAINS

Let's start with an EXTREMELY STUPID way to play the stock market: let's say we are only interested in the shares of a single company, TROOGLE™.

Q How do we know whether the share price will go UP (i.e.; time to buy) or DOWN (sell?) tomorrow?

NAÏVE MODEL | If TROOGLE went "up" today, it will go up tomorrow; Similarly for "down".

This is a REALLY BAD idea for obvious reasons. But the study of "historical stock prices" and their use in predicting future values is fundamental even today. A slightly better model involves using MORE historical data to infer probabilities: eg:



So, you'd come up with this graph if over the last (many) many days, TROOGLE had exhibited the following behavior:

- If the stock went up on day "n"
  - → Then it went up on day (n+1) 80% of the time
  - → Then it went down on day (n+1) 20% of the time
- If the stock went down on day "n"
  - → Then it went down on day (n+1) 90% of the time
  - → Then it went up on day (n+1) 10% of the time

Let's model this situation with a MATRIX (what else?):

$$A = \begin{array}{c|c} & \begin{matrix} \text{up} & \text{dn} \leftarrow \text{day } n \end{matrix} \\ \begin{matrix} \text{up} \\ \text{dn} \\ \uparrow \\ \text{day } (n+1) \end{matrix} & \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix} \end{array}$$

Def An $n \times n$ matrix is "STOCHASTIC" or "MARKOV" if:

- Every entry is ≥ 0
- Every column adds up to $\underline{1}$

The long-term behavior of TROGGLE depends on the behavior of A^k as k gets large...

$\boxed{\text{HERE'S WHY}}$:

Start with any "initial distribution" of probabilities: let's say FROOGLE went "up" on 4 and "down" on 6 of the last 10 days, so the initial probability is:

$$p_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

- What happens tomorrow?

$$p_1 = A p_0 = \begin{array}{c|cc} & u & d \\ \hline u & .8 & .1 \\ d & .2 & .9 \end{array} \begin{bmatrix} .4 \\ .6 \end{bmatrix}\begin{array}{c} u \\ d \end{array} = \begin{bmatrix} .38 \\ .62 \end{bmatrix}\begin{array}{c} u \\ d \end{array}$$

So: up with probability $\boxed{0.38}$ down with $\boxed{0.62}$!

- And the day after?

$$p_2 = A p_1 = A(Ap_0) = A^2 p_0$$

- And many days hence? $p_k = A p_{k-1} = \underline{A^k p_0}$

The sequence $(p_0, p_1, \ldots, p_k, \ldots)$ is called a MARKOV CHAIN associated to the stochastic/markov matrix A.

TWO THINGS:

- A cooler prediction model keeps track of longer "chains" in the history of the stock price:

  UU, UD, DU, DD ← Predicts 2 days ahead

  This gives a 4×4 stochastic matrix...

  Similarly, UUU, UUD, UDU, ..., DDD gives us 8×8 etc. ← Predicts 3!

☺ In order to decide long-term investment in TROOGLE, you'd probably want to know A^k for a gigantic k: this is where we can use EIGENVALUES & EIGENVECTORS: if A is diagonalizable, then we have

$$A^k = SD^kS^{-1},$$

$$\text{so} \quad P_k = A^kP_0 = SD^kS^{-1}P_0$$

Q: What happens when $k \to \infty$?? (does P_k stabilize?)

Def Let A be a stochastic matrix, and let $(P_0, P_1, \dots, P_k, \dots)$ be a Markov chain associated to A. Its $\boxed{\text{STEADY STATE}}$ is defined to be

$$P_\infty = \lim_{k\to\infty} P_k, \quad (\text{if it exists}).$$

Note If this steady state exists, then:

$$\begin{aligned} AP_\infty &= A \cdot \lim_{k\to\infty} P_k \\ &= A \cdot \lim_{k\to\infty} A^kP_0 \\ &= \lim_{k\to\infty} A^{k+1}P_0 \\ &= P_\infty \end{aligned}$$

So, $AP_\infty = P_\infty$: so, P_∞ is FIXED by A.

OR, P_∞ is an eigenvector of A with eigenvalue $= \underline{1}$.

MAIN QUESTIONS: When does a Steady State exist? And when does it depend on the initial state P_0?

The "TROGLE" matrix $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ has a unique steady state: just compute its eigenstuff:

EIGENVALUES

$\det(A - \lambda I) = 0$, so $(.8-\lambda)(.9-\lambda) - .02 = 0$
which means $\lambda^2 - 1.7\lambda + .7 = 0$, or
$(\lambda - 1)(\lambda - .7) = 0$.
So, $\lambda_1 = 1$ and $\lambda_2 = 0.7$.

EIGENVECTORS

v_1 in $N(A - I)$,
$A - I = \begin{bmatrix} -.2 & .1 \\ .2 & -.1 \end{bmatrix}$,
so $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ works,

v_2 in $N(A - 0.7I)$
$A - 0.7I = \begin{bmatrix} .1 & .1 \\ .2 & .2 \end{bmatrix}$
So $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ works,

but: want the components to add up to 1, so use
$v_1 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ ⟵ THIS is the P_∞: it is the STEADY STATE for all Markov chains of A.

Thm If A is a stochastic matrix with

- "largest" eigenvalue : $\lambda_1 = 1$,
- all other eigenvalues : $|\lambda| < 1$ (strictly less)

And if p_0 is a positive vector whose components add to 1, then

- $p_\infty = \lim_{k\to\infty} A^k p_0$ is an eigenvector for λ_1
- p_∞ is independent of the choice of p_0

Pf We will only do this for our $A = \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}$, the general case is similar!

So, $A = SDS^{-1}$, where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & .7 \end{bmatrix} \quad \text{with } |.7| < 1.$$

$$S = \begin{bmatrix} 1/3 & 1 \\ 2/3 & -1 \end{bmatrix} \quad \text{with the elements of } v_1 \text{ adding to } 1.$$

$$S^{-1} = \begin{bmatrix} 1 & 1 \\ 2/3 & -1/3 \end{bmatrix}$$

Now, $p_\infty = \lim_{k\to\infty} A^k p_0 = \lim_{k\to\infty} SD^kS^{-1}p_0 = S\left[\lim_{k\to\infty} D^k\right]S^{-1}p_0$

If $p_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ with $a+b=1$, then

$$p_\infty = \begin{bmatrix} 1/3 & 1 \\ 2/3 & -1 \end{bmatrix} \left(\lim_{k\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & .7 \end{bmatrix}^k \right) \begin{bmatrix} 1 & 1 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

BUT $\lim\limits_{k\to\infty} \begin{bmatrix} 1^k & 0 \\ 0 & .7^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

because $\lambda^k = 0$ whenever $|\lambda| < 1$,

So $$P_\infty = \begin{bmatrix} 1/3 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a+b \\ 2/3\,a - 1/3\,b \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} a+b \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3\,(a+b) \\ 2/3\,(a+b) \end{bmatrix} = (a+b) \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

BUT $a+b = 1$ by assumption on P_0,

So $P_\infty = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$, the eigenvector of A corresponding to $\boxed{\lambda_1 = 1}$

WARNING || We really need $|\lambda| < 1$ for all other eigenvalues! For example:

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \underline{\underline{-1}}$

And no steady state when $P_0 = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$:

$P_1 = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$, $P_3 = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$, -

alternates forever no P_∞ can be the limit!